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A COROLLARY TO KODAIRA–SPENCER'S THEOREM ON CONTINUITY OF EIGENVALUES

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ABSTRACT

We give an elementary proof of continuity of the determinant in the parameter for a smooth family of laplacians (of the same nullity) on a smooth family of holomorphic vector bundles over a compact complex manifold. Families of unitary flat bundles over a compact Riemann surface are discussed, as an example.

Introduction

Kodaira and Spencer introduced around 1960 the notion of differential families F of holomorphic vector bundles F_p over a compact complex manifold X, for p a parameter varying in an open subset U of a Euclidean space ([2], p. 324). Suppose X has a given Riemannian metric. Then Kodaira–Spencer define also differentiable families of hermitian metrics h_p on F_p and of associated laplacians Δ_p acting on the space of C^∞ sections of $F_p \to X, C^\infty(F_p)$. Each Δ_p has a spectrum of the form ([2], p. 351)

$$0 \le \lambda_1(p) \le \lambda_2(p) \le \cdots \le \lambda_m(p) \le \cdots, \quad \lambda_m(p) \to \infty \quad \text{as } m \to \infty.$$

Kodaira and Spencer showed the continuity of each eigenvalue λ_m in $p \in U$ ([2]), Theorem 7-2, 7-7) (see Section 1 for details).

It is natural to ask if the determinant of Δ_p defined by zeta function regularisation is continuous in p. Under the assumption that the dimension of the kernel

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of Δ_p is constant over U we prove in Proposition 1, Section 1 below that det Δ_p is continuous in p. In Section 2 we discuss the example of families of unitary flat holomorphic vector bundles over a compact Riemann surface of genus > 1. In this case continuity of the determinant is readily deduced also from the results in [7].

1. Continuity of det Δ_p

We recall the definitions first. Let X be a compact complex manifold of complex dimension n and U an open subset of $\mathbb{R}^N$.

Suppose $F \to X \times U$ is a $\mathcal{C}^\infty$ complex vector bundle of rank r. Then for each $p \in U$, the restriction of F to $X \times (p)$ is a smooth ($\mathcal{C}^\infty$) complex vector bundle $F_p \to X \times (p)$ of the same rank as F.

F or (F_p) is called a differentiable family of holomorphic vector bundles of rank r over X if there exist local trivializations of F

$$\pi^{-1}(U_j \times U) = \mathbf{C}^r \times U_j \times U$$

such that the transition functions

$$(\zeta_j, z, p) \to (\zeta_k, z, p)$$

are holomorphic in $z \in U_j$ and $\mathcal{C}^\infty$ in $p \in U$.

In particular, for a given $p \in U$ one has thus local trivializations of F_p. The fibre coordinates ζ_j thus obtained are called admissible fibre coordinates.

Suppose we are given a smooth function ψ_p of F_p for each $p \in U$. One says that ψ_p is $\mathcal{C}^\infty$ differentiable in p if each admissible fibre coordinate of $\psi_p(z)$ is a $\mathcal{C}^\infty$ function of (z, p).

Suppose we are given a linear operator $L_p : \mathcal{C}^\infty(F_p) \to \mathcal{C}^\infty(F_p)$ for each $p \in U$. $(L_p)_{p\in U}$ is called a differentiable family of linear operators if $L_p\psi_p$ is $\mathcal{C}^\infty$ differentiable in p whenever $\psi_p \in \mathcal{C}^\infty(F_p)$ is so. If each L_p is a linear differential operator, then $(L_p)_{p\in U}$ is a differentiable family if and only if in the admissible local trivializations one has

$$(L_p\psi_p)(z) = (\phi^1(z,p), \ldots, \phi^r(z,p))$$

where $\phi^\lambda(z,p) = \Sigma_{\mu=1}^{p} L_\mu^\lambda(z, p, \partial/\partial x_i, \partial/\partial y_i)\psi_p^\mu(z)$,

$$z = (z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n)$$

with L^p_μ polynomials in the $\partial/\partial x_i, \partial/\partial y_j$ with the coefficients C^∞ in (z,p).

Let h be an hermitian metric on F given in admissible coordinates over U_j by $\Sigma_i^\infty h_{j\lambda\mu}(z,p)\zeta_j^\lambda\bar{\zeta}_j^\mu$. In particular the $h_{j\lambda\mu}$ are smooth in (z,p).

Suppose the base manifold X has a Riemannian metric $g_{\alpha,\beta}$ and an associated volume element dv, then for $\psi_p, \phi_p \in C^\infty(F_p)$ there is an inner product defined by

$$(\psi_p, \phi_p)_p = \int_X (\Sigma h_{j\lambda\mu}(z,p)\psi^\lambda_{pj}(z)\bar{\phi}^\mu_{pj}(z))dv.$$

Let Δ_p be the laplacian of the hermitian metric h_p on F_p induced by h. Then $(\Delta_p)_{p\in U}$ form a differentiable family of linear elliptic differential operators of order 2. Each Δ_p is formally self-adjoint and strongly elliptic for the inner product $(\ ,\ )_p$. It is known that the spectrum of Δ_p is discrete and has the form

$$0 \le \lambda_1(p) \le \lambda_2(p) \le \cdots \le \lambda_m(p) \le \cdots, \quad \lambda_m(p) \to \infty \quad \text{as } m \to \infty.$$

KODAIRA–SPENCER'S THEOREM ([2], Theorems 7.2, 7.3, 7.7, 7.8.): *Let X be a compact complex manifold of complex dimension n with a Riemannian metric g. Let U be an open subset of $\mathbb{R}^N$. Let F be a differentiable family of holomorphic vector bundles F_p of rank r over $x, p \in U$. Let h be a smooth hermitian metric on F and let $(\Delta_p)_{p\in U}$ be the differentiable family of laplacians corresponding to the (h_p). Then*

(i) *each eigenvalue $\lambda_m(p)$ is continuous in p,*

(ii) *given $p_0 \in U$ there exists a small neighbourhood $N(p_0)$ of p_0 such that for each $p \in N(p_0)$*

$$\dim\ker\Delta_p \le \dim\ker\Delta_{p_0} < \infty.$$

Now the determinant of Δ_p is defined as follows. First set, for $s \in \mathbb{C}$,

$$\zeta_p(s) = 1/\Gamma(s)\int_0^\infty t^{s-1}(\Sigma e^{-\lambda_m(p)t} - m_0(p))dt;$$

ζ_p is analytically continued to zero ([5], Theorem 13.1). Define the determinant of Δ_p by

$$\det\Delta_p = |\exp -\zeta_p'(0)|.$$

PROPOSITION 1: *In addition to Kodaira–Spencer's hypotheses assume that* $\dim\ker\Delta_p$ *is the same number* h_0 *for all* $p \in U$. *Then* $\zeta_p'(0)$ *is continuous in* $p \in U$ *and so* $\det \Delta_p$ *is continuous in* p.

Proof: STEP 1: We first prove the continuity of $p \to \zeta_p'(s)$ for $\mathrm{Re}(s) \gg 0$.

Choose $p_0 \in U$ and let K be a (small) compact neighbourhood of p_0. It suffices to show that $\Sigma e^{-\lambda_m(p)t}$ is continuous in $p \in K$. We show that the series is uniformly convergent in p and invoke the continuity of each λ_m in $p \in K$. Set

$$\mu_m = \inf_{p\in K} \lambda_m(p), \quad m = h_0 + 1, h_0 + 2, \dots .$$

Each μ_m is positive by our hypothesis. The preceding series is majorised by

$$(*) \qquad \Sigma e^{-\mu_m t}.$$

Thus if $(*)$ converges uniformly for $t > 0$, then by Weierstrass's M-test the earlier series converges uniformly in $p \in K$.

To verify uniform convergence of $(*)$ to the right of zero, it suffices by Widder ([8], Theorem 3.3, pp. 47–48) to check that

$$\operatorname*{Lim}_{m\to\infty} (\log m/\mu_m) = 0.$$

For this we use the asymptotic behaviour of $\lambda_m(p)$. It is known ([5], p. 291) that

$$(m/\alpha(p)(\lambda_m(p))^n) \to 1 \quad \text{as } m \to \infty$$

with

$$\alpha(p) = (1/(2n)(2\pi)^{2n}) \int_X \int_{|\varepsilon'|=1} \mathrm{Trace}\,[a_2(z,\varepsilon',p)^{-n}]d\varepsilon' dv,$$

where the inner integration is with respect to the natural measure on the unit sphere of T^*X given by the symplectic structure of T^*X. By the strong ellipticity of Δ_p, the integrand is everywhere positive. One sees that the integrand is continuous in p, because the principal symbol a_2 is even smooth in p, by hypothesis. Thus $\alpha(p)$ is seen to be continuous in p. Thus there exist positive constants c_1, c_2 such that $c_1 \le \alpha(p) \le c_2$ for all $p \in K$. Thus

$$\begin{aligned} 0 = [\operatorname*{Lim}_{m\to\infty} (\log m/m^{1/n})(c_2)^{1/n}] &\le [\lim_{m\to\infty} \log m/\mu_m)] \\ &\le [\operatorname*{Lim}_{m\to\infty} ((\log m)(c_1)^{1/n}/m^{1/n})] = 0. \end{aligned}$$

Thus the middle term is zero and we are done.

STEP 2: Having verified that $p \to \zeta_p'(s)$ is continuous in p for $\mathrm{Re}(s) \gg 0$ we turn our attention to proving the same for $p \to \zeta_p'(0)$. For this purpose we consider how $\zeta_p(s)$ is defined in a neighbourhood of 0. To begin with the integral ([0], p. 79)

$$\int_1^\infty t^{s-1}\Big(\sum_{\lambda_n>0} e^{-\lambda_n(p)t}\Big)dt = r(s,p)$$

is an entire function in s. $r(s,p)$ has derivative at 0 continuous in p.

For $0 < t < 1$ consider the relation ([0], p. 79)

$$\sum_{\lambda_n>0} e^{-\lambda_n(p)t} = \sum_{k \le n_0} t^{(k-m)/2} a_k(p) + O(t^{(n_0-m)/2})$$

where $a_k(p) = \int_X a_k(x, \Delta_p) d\,\mathrm{vol}$, $m = \dim_{\mathbb{R}} X$ and $a_k(x, \Delta_p)$ is a local scalar invariant of the jets of the total symbol of Δ_p. Thus $a_k(p)$ is continuous in p, because the total symbol varies continuously in p.

We consider the error term $E(t,p)$. Let p vary in a compact ball K of positive radius. By an application of Garding's inequality the kernel $K_p(t,x,x)$ of $e^{-t\Delta_p}$ is continuous on $K \times (0,\infty) \times X$. Thus the bound ([0], p. 54)

$$\|K_p(t,x,x) - \sum_{n=0}^{n(k)} t^{(n-d)/2} a_n(x,p)\|_{\infty,k} < Ct^k$$

is uniform in p for k small enough, $n(k)$ being the same integer for all p in K. This yields a uniform bound for the error term $E(t,p)$ (which is the trace of the above difference and so is continuous).

Choose k large enough and use the Dominated Convergence Theorem to derive the continuity of $\int_0^1 t^{s-1} E(t,p)dt$. The continuity of the derivative at $s = 0$ follows from standard criteria.

The integral is analytic on $\{\mathrm{Re}(s) > -(n_0 - m)/2\}$. Thus considering the expansion ([0], p. 79) near the origin

$$\Gamma(s)\zeta_p(s) = \sum_{n \le n_0} 2(2s + n - m)^{-1} a_n(p) + r_{n_0}(s,p)$$

(with $r_{n_0}(s,p)$ analytic in $s \in \{\,\mathrm{Re}(s) > -(n_0 - m)/2\}$ with derivative continuous in p) we have $p \to \zeta_p'(0)$ is continuous. ∎

2. An Example

Let X be a compact Riemann surface of genus $g > 1$ endowed with the canonical (Poincaré) metric. Narasimhan and Seshadri ([3]) considered the space

$$M_n = \{\text{equivalence classes } \tilde{\rho} \text{ of irreducible unitary representations } \rho : \pi_1(X) \to U(n) \text{ of dimension } n \text{ of } \pi_1(X)\}.$$

They showed that M_n is a complex manifold of complex dimension $n^2(g-1)-1$.

Let U be a (small) open subset of M_n. It is known that the associated holomorphic vector bundles F_ρ over x determined by $\tilde{\rho} \in U$ give a differentiable family of homomorphic vector bundles ([3], Remark on p.80). Further, the natural flat metrics form a differentiable family too ([6], Remark (iv), p. 17). One notes that equivalent unitary representations ρ, ρ' correspond to isometrically isomorphic flat bundles which have thus the same spectrum for their laplacians $\Delta_\rho, \Delta_{\rho'}$. Thus the eigenvalues $\lambda_m(\tilde{\rho})$ are functions defined on the whole of M_n.

Remark 1: By Kodaira–Spencer's Theorem each λ_m is continuous on M_n (for each n). ∎

Remark 2: For $n = 1, M_n = \text{Pic}(X)$, for the trivial character $\tau \in \text{Pic}(X), \tau \equiv 1, \Delta_\tau = \Delta_X$, the Laplace–Beltrami operator of X, 0 is an eigenvalue of Δ_p for $\tilde{\rho} \in \text{Pic}(X)$ if and only if $\tilde{\rho} = \tau$ ([11], p. 353). ∎

Remark 3: For $n > 1, \tilde{\rho} \in M_n$, 0 is never an eigenvalue of Δ_p ([11], p. 353). ∎

Remark 4: $\det \Delta_p$ is continuous on M_n for $n > 1$ and on $\text{Pic}(X) \setminus (\tau)$ by Proposition 1 above. ∎

Remark 5: Remark 4 can be deduced also from [7]. There it was shown that the evaluation of the Selberg Zeta Function at 1

$$Z_\Gamma(1,.) : \tilde{\rho} \to Z_\Gamma(1, \tilde{\rho})$$

is continuous on M_n for each n. For nontrivial $\tilde{\rho}_1, \tilde{\rho}_2$ the formula of Ray–Singer ([4], [7]) says

$$\det \Delta_{\tilde{\rho}_1} / \det \Delta_{\tilde{\rho}_2} = Z_\Gamma(1, \tilde{\rho}_1)/Z_\Gamma(1, \tilde{\rho}_2).$$

Fixing ρ_2 and varying ρ_1 one has the continuity of $\det \Delta_{\tilde{\rho}_1}$ in $\tilde{\rho}_1$ on M_n if $n > 1$ and $\text{Pic}(X) \setminus (\tau)$ if $n = 1$. ∎

Remark 6: det $\Delta_{\bar{\rho}}$ as defined in Section 1 is not continuous at the trivial character $\tau \in \mathrm{Pic}(X)$, for $Z_\Gamma(1,\tau) = 0$ ([1], p. 72), while by definition the determinant is positive. ∎

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